Discrete Applied Mathematics 158 (2010) 951–963

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Combinatorial constructions of fault-tolerant routings with levelled minimum optical indices

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ARTICLE INFO

Article history:
Received 17 March 2009
Received in revised form 2 November 2009
Accepted 17 December 2009
Available online 4 January 2010

Keywords:
Candelabra $\overrightarrow{P_3}$-system
Fault tolerance
Large set of $\overrightarrow{P_3}$-design
Levelled minimum optical indices
Optical network
Routing

ABSTRACT

The design of fault-tolerant routings with levelled minimum optical indices plays an important role in the context of optical networks. However, not much is known about the existence of optimal routings with levelled minimum optical indices besides the results established by Dinitz, Ling and Stinson via the partitionable Steiner quadruple systems approach. In this paper, we introduce a new concept of a large set of even levelled $\overrightarrow{P_3}$-design of order v and index 2, denoted by $(v, \overrightarrow{P_3}, 2)$-LELD, which is equivalent to an optimal, levelled $(v-2)$-fault-tolerant routing with levelled minimum optical indices of the complete network with v nodes. On the basis of the theory of three-wise balanced designs and partitionable candelabra systems, several infinite classes of $(v, \overrightarrow{P_3}, 2)$-LELDs are constructed. As a consequence, the existence problem for optimal routings with levelled minimum optical indices is solved for nearly a third of the cases.

1. Introduction

The design of routings in optical networks has been a topic of considerable recent interest (see, for examples, [1–4,15]). In the model of *WDM optical networks*, namely, wavelength division multiplexing optical networks, routing nodes are joined by fiber-optic links, and each link can support some fixed number of wavelengths. Each routing path uses a particular wavelength, and two paths must use different wavelengths if they have common links. Most research concentrates on determining the minimum total number of wavelengths used in the network, which is related to two basic invariants — the *arc-forwarding* and *optical indices*. The *f-tolerant arc-forwarding* and *f-tolerant optical indices* were introduced by Maňuch and Stacho when they considered the fault-tolerant issues in [15]. The parameter f represents the number of faults that can be tolerated in the optical network. That is, we can provide a routing between any two nodes even if some number (up to f) of nodes and/or links fail. In this paper, we focus on the fault-tolerant routings in the complete optical network.

We first review definitions of several desirable properties that we are going to investigate in the setting of fault-tolerant routings. These terms have previously been defined in papers such as [3,7,8].

Let $G = (V(G), A(G))$ be a *symmetric* directed graph, i.e., $(u, v) \in A(G)$ implies $(v, u) \in A(G)$. An *f-fault-tolerant routing* is a set of directed paths in G, say $\mathcal{R}_f = \{P_i(u, v) : u \neq v, 0 \leq i \leq f\}$, where the following two properties are satisfied:

1. every $P_i(u, v)$ is a directed path in G from vertex u to vertex v, and
2. for all vertices u and v where $u \neq v$, the $f + 1$ paths $P_i(u, v)$ $(0 \leq i \leq f)$ are internally vertex disjoint.

For $0 \leq i \leq f$, define $\mathcal{L}_i = \{P_i(u, v) : u \neq v\}$, which is called the ith *level* of the routing. For convenience, we write $\mathcal{R}_f$ in the form $\mathcal{R}_f = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_f)$. It is clear that $\mathcal{R}_j = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_j)$ is a j-fault-tolerant routing, for $0 \leq j \leq f$. Therefore an f-fault-tolerant routing can be regarded as a sequence of j-fault-tolerant routings for $0 \leq j \leq f$, namely, $(\mathcal{R}_0, \ldots, \mathcal{R}_f)$.

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doi:10.1016/j.dam.2009.12.008

The *load* $\overrightarrow{\pi}(e)$ on an arc $e \in A(G)$ is defined to be the number of paths in the routing that contain the arc e. Define $\overrightarrow{\pi}(\mathcal{R}_f) = \max\{\overrightarrow{\pi}(e) : e \in A(G)\}$. Further, define $\overrightarrow{\pi_f}(G) = \min_{\mathcal{R}_f}\{\overrightarrow{\pi}(\mathcal{R}_f)\}$ and call $\overrightarrow{\pi_f}(G)$ the *f-fault-tolerant arc-forwarding index* of G. The routing $\mathcal{R}_f$ is said to be *optimal* if $\overrightarrow{\pi_f}(G) = \overrightarrow{\pi}(\mathcal{R}_f)$, and to be *optimal, levelled* if $\overrightarrow{\pi_j}(G) = \overrightarrow{\pi}(\mathcal{R}_j)$, for all $0 \le j \le f$.

Let $n \ge 2$ be a positive integer and let $\overrightarrow{K_n}$ denote the complete symmetric directed graph on a set of n vertices, say X. By [6–8], we have:

Theorem 1.1. *Suppose there is an f-fault-tolerant routing of $\overrightarrow{K_n}$, say $\mathcal{R}_f = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_f)$. Then $0 \le f \le n-2$ and $\overrightarrow{\pi_f}(\overrightarrow{K_n}) \ge 2f+1$ for all f, $0 \le f \le n-2$. Furthermore, equality is attained (i.e., the routing is an optimal, levelled routing) if and only if the following properties are satisfied:*

1. *$\mathcal{L}_0$ consists of all the arcs in $\overrightarrow{K_n}$ (that is, $\mathcal{L}_0$ comprises $n(n-1)$ directed paths, each having length 1), and*
2. *for $1 \le j \le n-2$, $\mathcal{L}_j$ consists of $n(n-1)$ directed paths, each having length 2, such that every arc in $\overrightarrow{K_n}$ is in exactly two directed paths in $\mathcal{L}_j$.*

The following theorem was proved in [6–8].

Theorem 1.2 ([6–8]). *For each integer $n \ge 2$, there exists an optimal, levelled $(n-2)$-fault-tolerant routing of $\overrightarrow{K_n}$.*

Let $\mathcal{W}$ be a set of wavelengths. A *wavelength assignment* to the directed paths in $\mathcal{R}_f$ is defined to be a map $\alpha : \mathcal{R}_f \to \mathcal{W}$ such that $\alpha(P) \ne \alpha(Q)$ whenever $P, Q \in \mathcal{R}_f$ are two directed paths that share a common arc. Let $\overrightarrow{w}(\mathcal{R}_f)$ denote the minimum cardinality of a set $\mathcal{W}$ such that an assignment of wavelengths for $\mathcal{R}_f$ exists that satisfies the previous property. Define $\overrightarrow{w_f}(G) = \min_{\mathcal{R}_f}\{\overrightarrow{w}(\mathcal{R}_f)\}$ and call $\overrightarrow{w_f}(G)$ the *f-fault-tolerant optical index* of G. It is obvious that $\overrightarrow{w_f}(G) \ge \overrightarrow{\pi_f}(G)$. An optimal, levelled f-fault-tolerant routing $\mathcal{R}_f$ is said to have *minimum optical indices* if $\overrightarrow{w}(\mathcal{R}_i) = \overrightarrow{w_i}(G)$ for all i such that $0 \le i \le f$.

For $0 \le i \le f$, construct a graph whose vertices are the directed paths in $\mathcal{L}_i$. Two vertices are defined to be *adjacent* if they have a common arc. This graph is called the *path graph* of $\mathcal{L}_i$. In many applications, it could be desirable that wavelength assignments for $\mathcal{R}_{i-1}$ do not change when we determine wavelength assignments for $\mathcal{R}_i$. Under this assumption, it is easy to see that we require at most δ_i "extra" wavelengths when we proceed with the assignment from $\mathcal{R}_{i-1}$ to $\mathcal{R}_i$, where δ_i is the chromatic number of the path graph of $\mathcal{L}_i$, for $0 \le i \le f$. Define $\overrightarrow{w_L}(\mathcal{R}_i) = \sum_{j=0}^{i}\delta_j$, $0 \le i \le f$. It is clear that $\overrightarrow{w}(\mathcal{R}_i) \le \overrightarrow{w_L}(\mathcal{R}_i)$. An optimal, levelled f-fault-tolerant routing $\mathcal{R}_f$ is said to have *levelled minimum optical indices* if $\overrightarrow{w_L}(\mathcal{R}_i) = \overrightarrow{w_i}(G)$ for all i such that $0 \le i \le f$. A routing having levelled minimum optical indices has minimum optical indices. The converse is not true. Here is a counterexample given in [6].

Example 1.3 ([6]). The unique one-fault-tolerant routing of $\overrightarrow{K_3}$ has minimum optical indices, but it does not have levelled minimum optical indices.

Proof. The unique one-fault-tolerant routing $\mathcal{R}_1$ of $\overrightarrow{K_3}$ on $X = \{0, 1, 2\}$ is as follows:

$$\mathcal{L}_0 : (0,1)^3, (0,2)^2, (1,0)^1, (1,2)^2, (2,0)^1, (2,1)^3$$
$$\mathcal{L}_1 : (0,2,1)^1, (0,1,2)^1, (1,2,0)^3, (1,0,2)^3, (2,1,0)^2, (2,0,1)^2$$

Here, the superscripts denote wavelengths. It is clear that $\overrightarrow{w_L}(\mathcal{R}_0) = \overrightarrow{w}(\mathcal{R}_0) = 1$. The wavelength assignment shows that $\overrightarrow{w}(\mathcal{R}_1) = 3$ which is minimal. However, when we proceed with the assignment from $\mathcal{R}_0$ to $\mathcal{R}_1$, three "extra" wavelengths should be used. That is $\overrightarrow{w_L}(\mathcal{R}_1) = 4$. Therefore, the routing $\mathcal{R}_1$ does not have levelled minimum optical indices. □

Let $\mathcal{R}_f = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_f)$ be an optimal, levelled f-fault-tolerant routing of $\overrightarrow{K_n}$ described in Theorem 1.1. The path graph of $\mathcal{L}_0$ contains no edges, so $\delta_0 = 1$. For each i, $1 \le i \le f$, each directed path of $\mathcal{L}_i$ has exactly two arcs and each arc in $\overrightarrow{K_n}$ occurs in two directed paths. Thus the path graph of $\mathcal{L}_i$ is a union of disjoint cycles. It is straightforward that $\overrightarrow{w_L}(\mathcal{R}_i) \ge \overrightarrow{\pi_i}(\overrightarrow{K_n}) \ge 2i+1$; equality holds when $\delta_i = 2$ for all $1 \le i \le f$, which happens if and only if all the cycles have even length. So we have:

Theorem 1.4. *An optimal, levelled $(n-2)$-fault-tolerant routing of $\overrightarrow{K_n}$, say $\mathcal{R}_{n-2} = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{n-2})$, has levelled minimum optical indices if and only if the following property is satisfied:*

3. *The path graph of each $\mathcal{L}_i$ $(1 \le i \le n-2)$ consists of even cycles.*

On the basis of the theory of three-wise balanced designs and partitionable candelabra systems, Ji [12] gave a simple new proof for the existence of large sets of Steiner triple systems. In this paper, via a similar approach we will concentrate on constructing optimal, levelled $(n-2)$-fault-tolerant routings $\mathcal{R}_{n-2}$ with levelled minimum optical indices of the complete directed graph $\overrightarrow{K_n}$. The following results are known.

Theorem 1.5 ([6]). *For each n, $5 \le n \le 8$, $n = 4^k$ or $n = 2(p^k+1)$ with $p \in \{7, 31\}$ and $k \ge 1$, there exists an optimal, levelled $(n-2)$-fault-tolerant routing of $\overrightarrow{K_n}$ that has levelled minimum optical indices.*

The paper is organized as follows. In Section 2, we first define a new class of combinatorial objects, large sets of even levelled $(n, \overrightarrow{P_3}, 2)$-design (LELDs), which are equivalent to the optimal, levelled $(n-2)$-fault-tolerant routings with levelled minimum optical indices of the complete network with n nodes. Then, we present a recursive construction for LELDs by using the theory of three-wise balanced designs and partitionable candelabra systems. In Section 3, some small ingredient designs are constructed directly. Combining these ingredient designs together with the recursive methods established in Section 2, we are able to give several infinite classes of LELDs in Section 4, which imply the existence of the corresponding routings having levelled minimum optical indices. Some concluding remarks are given in Section 5.

2. Definitions and recursive constructions

Let $\overrightarrow{P_3} = (a, b, c)$ be a directed path which contains two arcs (a, b) and (b, c). Let $\lambda\overrightarrow{K_n}$ be the directed multigraph on n vertices in which each ordered pair of vertices is joined by λ arcs. A $\overrightarrow{P_3}$*-decomposition* of $\lambda\overrightarrow{K_n}$ is a partition of the arcs of $\lambda\overrightarrow{K_n}$ into paths isomorphic to $\overrightarrow{P_3}$, which is also called a $\overrightarrow{P_3}$*-design* of order n and index λ and denoted by $(n, \overrightarrow{P_3}, \lambda)$*-design*. A similar concept of P_3-decomposition of the undirected graph was given in [14]. If a set $\mathcal{B}$ of $\overrightarrow{P_3}$ paths contains exactly one path from u to v for every ordered pair of vertices u and v in $\overrightarrow{K_n}$, then we call the set $\mathcal{B}$ a *level*. A level is said to be *even* if its path graph consists of even cycles. An $(n, \overrightarrow{P_3}, 2)$-design is said to be *levelled* (*even levelled*) if it is a level (an even level), which is denoted by $(n, \overrightarrow{P_3}, 2)$-LD ($(n, \overrightarrow{P_3}, 2)$-ELD).

A *large set of* $(n, \overrightarrow{P_3}, 2)$*-LD*, denoted by $(n, \overrightarrow{P_3}, 2)$-LLD, is a partition $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}$ of all $\overrightarrow{P_3}$ paths in $\overrightarrow{K_n}$ such that each $\mathcal{B}_i$ forms an $(n, \overrightarrow{P_3}, 2)$-LD. If each $\mathcal{B}_i$ is even levelled, then we call the partition a *large set* of $(n, \overrightarrow{P_3}, 2)$-ELD, which is denoted by $(n, \overrightarrow{P_3}, 2)$-LELD.

As a consequence of Theorems 1.1 and 1.4, we have the following theorem.

Theorem 2.1. *Suppose that* $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}$ *form an* $(n, \overrightarrow{P_3}, 2)$*-LLD. Let* $\mathcal{L}_0$ *consist of all arcs in* $\overrightarrow{K_n}$*,* $\mathcal{L}_i$ *consist of all paths in* $\mathcal{B}_i$*,* $1 \le i \le n-2$*; then* $\mathcal{R}_{n-2} = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{n-2})$ *is an optimal, levelled* $(n-2)$*-fault-tolerant routing of* $\overrightarrow{K_n}$*. The reverse is also true. Furthermore,* $\mathcal{R}_{n-2}$ *has levelled minimum optical indices if and only if* $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}$ *form an* $(n, \overrightarrow{P_3}, 2)$*-LELD.*

By Theorems 1.2 and 1.5, we have the following corollaries.

Corollary 2.2. *There exists an* $(n, \overrightarrow{P_3}, 2)$*-LLD for any integer* $n \ge 2$.

Corollary 2.3. *For each n,* $5 \le n \le 8$*,* $n = 4^k$ *or* $n = 2(p^k + 1)$ *with* $p \in \{7, 31\}$*, there exists an* $(n, \overrightarrow{P_3}, 2)$*-LELD.*

In the remainder of this section, we will present a recursive construction for LELDs via partitionable candelabra systems having the even levelled property. First, we give some notation and terminology. The interested reader may refer to [5] for the undefined terms as well as a general overview of design theory.

2.1. Notation and terminology

Let v, s be two non-negative integers, t be a positive integer, and K be a set of positive integers. A *candelabra t-system* (or t-CS) of order v and block sizes from K, denoted by $\mathrm{CS}(t, K, v)$, is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements (called *points*);
(2) S is an s-subset (called the *stem* of the *candelabra*) of X;
(3) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets (called *groups* or *branches*) of $X \setminus S$, which partition $X \setminus S$;
(4) $\mathcal{A}$ is a collection of subsets (called *blocks*) of X, each of cardinality from K;
(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$, for all i, is contained in a unique block of $\mathcal{A}$, and no t-subset of $S \cup G_i$, for any i, is contained in any block of $\mathcal{A}$.

By the *group type* (or *type*) of a t-CS $(X, S, \mathcal{G}, \mathcal{A})$ we mean the list $(|G| \mid G \in \mathcal{G} : |S|)$ of group sizes and stem size, where the stem size is separated from the group sizes by a colon. If a t-CS has n_i groups of size g_i, $1 \le i \le r$, and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ to denote the group type.

Let $(X, S, \mathcal{G}, \mathcal{A})$ be a $\mathrm{CS}(3, K, v)$ of type $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ with $s > 0$ and let $S = \{\infty_1, \ldots, \infty_s\}$. For $1 \le i \le s$, let $\mathcal{A}_i = \{A \setminus \{\infty_i\} : A \in \mathcal{A}, \infty_i \in A\}$ and $\mathcal{A}_T = \{A \in \mathcal{A} : A \cap S = \emptyset\}$. Then the $(s+3)$-tuple $(X, \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_s, \mathcal{A}_T)$ is called an *s-fan design*. If the block sizes of $\mathcal{A}_i$ and $\mathcal{A}_T$ are from K_i $(1 \le i \le s)$ and K_T, respectively, then the s-fan design is denoted by s-$\mathrm{FG}(3, (K_1, K_2, \ldots, K_s, K_T), \sum_{i=1}^{r} n_i g_i)$ of type $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$.

A $\mathrm{CS}(t, K, v)$ of type $(1^v : 0)$ or of type $(1^{v-1} : 1)(X, S, \mathcal{G}, \mathcal{A})$ is usually called a *t-wise balanced design*. The notation $S(t, K, v)$ is often used and the design is called a *Steiner system*. An $\mathrm{S}(3, 4, v)$ is also called a *Steiner quadruple system* and denoted by $\mathrm{SQS}(v)$, whose necessary and sufficient condition for existence is $v \equiv 2, 4 \pmod 6$ [9]. It is well known that an $S(3, \{4, 6\}, v)$ exists if and only if $v \equiv 0 \pmod 2$ [10], and an $S(3, \{4, 5, 6\}, v)$ exists if and only if $v \equiv 0, 1, 2 \pmod 4$ and $v \ne 9, 13$ [11].

A *group divisible t-design* of order v with block sizes from K, denoted by GDD(t, K, v), is a triple $(X, \mathcal{G}, \mathcal{B})$ such that:

(1) X is a set of v elements;
(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets (called *groups*) of X which partition X;
(3) $\mathcal{B}$ is a family of subsets (called *blocks*) of X, each of cardinality from K such that each block intersects any given group in at most one point;
(4) every t-subset T of X from t distinct groups is contained in a unique block.

The *type* of the GDD(t, K, v) is defined as the list $(|G||G \in \mathcal{G})$. If a GDD has n_i groups of size g_i, $1 \le i \le r$, then we use the exponential notation $g_1^{n_1}g_2^{n_2}\cdots g_r^{n_r}$ to denote the group type. Mills in [16] used H(n, g, k, t) design to denote the GDD(t, k, ng) of type g^n. In this paper, we use H$(g_1^{n_1}g_2^{n_2}\cdots g_r^{n_r})$ to denote the GDD$(3, 4, \sum n_i g_i)$ of type $g_1^{n_1}g_2^{n_2}\cdots g_r^{n_r}$ for short.

Theorem 2.4 (*[13,16]*). *For $n > 3$ and $n \ne 5$, an* H(g^n) *exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3. For $n = 5$, an* H(g^n) *exists when g is even, $g \ne 2$ and $g \not\equiv 10, 26 \pmod{48}$.*

A *candelabra $\overrightarrow{P_3}$-system* of order n, denoted by $(n, \overrightarrow{P_3})$-CS, is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

1. X is the vertex set of $\overrightarrow{K_n}$;
2. S is a subset of X of size s;
3. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of $X \setminus S$, which partition $X \setminus S$;
4. $\mathcal{A}$ consists of all $\overrightarrow{P_3}$ paths of $\overrightarrow{K_n}$ not contained in any subgraph spanned by $S \cup G$ for each $G \in \mathcal{G}$.

A *group divisible $\overrightarrow{P_3}$-design* of order n and index λ, denoted by $(n, \overrightarrow{P_3}, \lambda)$-GDD, is a triple $(X, \mathcal{G}, \mathcal{B})$ such that:

1. X is the vertex set of $\overrightarrow{K_n}$;
2. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of X which partition X;
3. $\mathcal{B}$ is a family of $\overrightarrow{P_3}$ paths of $\overrightarrow{K_n}$ such that each path intersects any given group in at most one point;
4. each arc from two different groups is contained in exactly λ paths of $\mathcal{B}$.

For the *group type* of an $(n, \overrightarrow{P_3})$-CS and an $(n, \overrightarrow{P_3}, \lambda)$-GDD, we use the same notation as the group type of t-CS and GDD(t, K, v) respectively. An $(n, \overrightarrow{P_3}, 2)$-GDD $(X, \mathcal{G}, \mathcal{B})$ is called a *level*, denoted by $(n, \overrightarrow{P_3}, 2)$-LGDD, if $\mathcal{B}$ contains exactly one path from u to v, for every ordered pair of vertices u, v from two different groups. An $(n, \overrightarrow{P_3}, 2)$-GDD is called *even levelled*, denoted by $(n, \overrightarrow{P_3}, 2)$-ELGDD, if it is an even level, that is its path graph consists of even cycles.

An $(n, \overrightarrow{P_3})$-CS of type $(g_1^{a_1}g_2^{a_2}\ldots g_r^{a_r} : s)$ $(X, S, \mathcal{G}, \mathcal{A})$ with $s \ge 2$ is called *partitionable*, denoted by $(n, \overrightarrow{P_3})$-PCS, if the path set $\mathcal{A}$ can be partitioned into components $\mathcal{A}_x$ $(x \in G, G \in \mathcal{G})$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{s-2}$ with the following two properties: (i) for any $x \in G$ and $G \in \mathcal{G}$, $\mathcal{A}_x$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-GDD of type $1^{n-s-|G|}(|G|+s)^1$ with $G \cup S$ as the long group; (ii) for $1 \le i \le s-2$, $(X \setminus S, \mathcal{G}, \mathcal{A}_i)$ is an $(n-s, \overrightarrow{P_3}, 2)$-GDD of type $g_1^{a_1}g_2^{a_2}\ldots g_r^{a_r}$. If each component of an $(n, \overrightarrow{P_3})$-PCS is levelled (even levelled), then we denote it as $(n, \overrightarrow{P_3})$-LPCS ($(n, \overrightarrow{P_3})$-ELPCS).

In order to use an $(n, \overrightarrow{P_3})$-ELPCS to construct an $(n, \overrightarrow{P_3}, 2)$-LELD, we need a holey large set. Let X be an n-element set and Y be an s-subset of X with $s \ge 2$. Let $\overrightarrow{X}^{(3)}$ and $\overrightarrow{Y}^{(3)}$ denote all $\overrightarrow{P_3}$ paths in the complete symmetric directed graph on X and Y, respectively. A *holey large set of $(n, \overrightarrow{P_3}, 2)$-LD* on X with a hole Y, denoted by $(n, s; \overrightarrow{P_3}, 2)$-HLLD, is a partition of $\overrightarrow{X}^{(3)} \setminus \overrightarrow{Y}^{(3)}$ into components $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n-2}$ with the properties that (1) for $1 \le i \le n-s$, each $(X, \mathcal{A}_i)$ is an $(n, \overrightarrow{P_3}, 2)$-LD; (2) for $n-s+1 \le i \le n-2$, each $\mathcal{A}_i$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-LGDD of type $1^{n-s}s^1$ with the long group Y. If each component of an $(n, s; \overrightarrow{P_3}, 2)$-HLLD is even, then we denote it by $(n, s; \overrightarrow{P_3}, 2)$-HLELD.

A *generalized $\overrightarrow{P_3}$-frame*, denoted by F$(\overrightarrow{P_3}, v\{m\})$, is a collection of triples $\{(X, \mathcal{G}, \mathcal{B}_r) : r \in X\}$, where X is the vertex set of $\overrightarrow{K_{vm}}$, $\mathcal{G}$ is a partition of X into v sets of m points each, such that $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{B}_r)$ is a $((v-1)m, \overrightarrow{P_3}, 2)$-GDD of type m^{v-1} for each $r \in G$ and $G \in \mathcal{G}$, $\cup_{r \in X} \mathcal{B}_r$ consists of all the $\overrightarrow{P_3}$ paths intersecting every given group in at most one point, and all $\mathcal{B}_r$, $r \in X$ are pairwise disjoint. If each component of an F$(\overrightarrow{P_3}, v\{m\})$ is levelled (even levelled), then we denote it by LF$(\overrightarrow{P_3}, v\{m\})$ (ELF$(\overrightarrow{P_3}, v\{m\})$).

2.2. Recursive constructions

Theorem 2.5. *Suppose there exists an* H(g^n) *and an* ELF$(\overrightarrow{P_3}, 4\{m\})$*; then there exists an* ELF$(\overrightarrow{P_3}, n\{gm\})$.

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be an H(g^n). Let $X' = X \times Z_m$ and $\mathcal{G}' = \{G' = G \times Z_m : G \in \mathcal{G}\}$. We will construct an ELF$(\overrightarrow{P_3}, n\{gm\})$ on X' with group set $\mathcal{G}'$.

For each $B \in \mathcal{B}$, construct an ELF$(\overrightarrow{P_3}, 4\{m\})$ on $B \times Z_m$ with group set $\{x \times Z_m : x \in B\}$. Denote as $\mathcal{A}_B$ the path set which can be partitioned into $4m$ subsets $\mathcal{A}_B(x, i)$, $(x, i) \in B \times Z_m$, such that each $\mathcal{A}_B(x, i)$ is a $(3m, \overrightarrow{P_3}, 2)$-ELGDD of type m^3 on $(B \setminus \{x\}) \times Z_m$ with group set $\{y \times Z_m : y \in B \setminus \{x\}\}$.

For each $x \in X$ and $i \in Z_m$, let $\mathcal{C}(x, i) = \bigcup_{x \in B \in \mathcal{B}} \mathcal{A}_B(x, i)$. It is easy to check that $\mathcal{C}(x, i)$ is a $(gm(n-1), \overrightarrow{P_3}, 2)$-ELGDD of type $(gm)^{n-1}$ with group set $\mathcal{G}' \setminus \{G' : x \in G\}$. In fact, since every two distinct blocks B and B' from the set $\{D \in \mathcal{B} : x \in D\}$ have at most one common point besides x, every two paths from $\mathcal{A}_B(x, i)$ and $\mathcal{A}_{B'}(x, i)$ respectively have no common arc. Then by the definition of path graph, $\mathcal{C}(x, i)$ is even.

Since all $\mathcal{C}(x, i)$ with $x \in X$ and $i \in Z_m$ are disjoint, they form an ELF$(\overrightarrow{P_3}, n\{gm\})$. □

Theorem 2.6. *Suppose there exists an $(n, \overrightarrow{P_3})$-ELPCS of type $(g_0^1 g_1^{a_1} g_2^{a_2} \dots g_r^{a_r} : s)$ with $n = \sum_{1 \le i \le r} a_i g_i + g_0 + s$. If there is a $(g_i + s, s; \overrightarrow{P_3}, 2)$-HLELD for $1 \le i \le r$, then there is an $(n, g_0 + s; \overrightarrow{P_3}, 2)$-HLELD. Furthermore, if a $(g_0 + s, \overrightarrow{P_3}, 2)$-LELD exists, then there is an $(n, \overrightarrow{P_3}, 2)$-LELD.*

Proof. Let $(X, S, \mathcal{G}, \mathcal{A})$ be the given $(n, \overrightarrow{P_3})$-ELPCS of type $(g_0^1 g_1^{a_1} g_2^{a_2} \dots g_r^{a_r} : s)$. By the definition, $\mathcal{A}$ can be partitioned into subsets $\mathcal{A}_y$ ($y \in G$ and $G \in \mathcal{G}$) and $\mathcal{A}_i$ ($1 \le i \le s-2$) with the properties that each $\mathcal{A}_y$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{n-|G|-s}(|G|+s)^1$ with the long group $G \cup S$ and that each $(X \setminus S, \mathcal{G}, \mathcal{A}_i)$ is an $(n-s, \overrightarrow{P_3}, 2)$-ELGDD of type $g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}$.

Let G_0 be the special group with $|G_0| = g_0$. For each $G \in \mathcal{G}$ with $G \ne G_0$, suppose the given $(|G|+s, s; \overrightarrow{P_3}, 2)$-HLELD consists of $|G|(|G|+s, \overrightarrow{P_3}, 2)$-ELDs with path sets $\mathcal{B}_y$ ($y \in G$) and $s-2$ $(|G|+s, \overrightarrow{P_3}, 2)$-ELGDDs of type $1^{|G|}s^1$ with the long group S and path sets $\mathcal{B}_i^G$ ($1 \le i \le s-2$).

For each $y \in G, G \in \mathcal{G}$ with $G \ne G_0$, let $\mathcal{C}_y = \mathcal{A}_y \bigcup \mathcal{B}_y$. Since the path graph of $\mathcal{C}_y$ is the disjoint union of path graphs of $\mathcal{A}_y$ and $\mathcal{B}_y$, each $(X, \mathcal{C}_y)$ is an $(n, \overrightarrow{P_3}, 2)$-ELD. For $1 \le i \le s-2$, let $\mathcal{C}_i = \mathcal{A}_i \bigcup (\bigcup_{G \in \mathcal{G}, G \ne G_0} \mathcal{B}_i^G)$. It is easy to check that the path graph of $\mathcal{C}_i$ is also the disjoint union of path graphs of all its components. Then each $\mathcal{C}_i$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{n-g_0-s}(g_0+s)^1$ with the long group $G_0 \cup S$. So $\{\mathcal{C}_y : y \in G \in \mathcal{G}, G \ne G_0\} \bigcup \{\mathcal{A}_y : y \in G_0\} \bigcup \{\mathcal{C}_i : 1 \le i \le s-2\}$ forms an $(n, g_0 + s; \overrightarrow{P_3}, 2)$-HLELD.

Finally, suppose the given $(g_0 + s, \overrightarrow{P_3}, 2)$-LELD on $G_0 \bigcup S$ has $g_0 + s - 2$ disjoint $(g_0 + s, \overrightarrow{P_3}, 2)$-ELDs with path sets $\mathcal{B}_y$ ($y \in G_0$) and $\mathcal{B}_i$ ($1 \le i \le s-2$), respectively. Then the $(X, \mathcal{A}_y \cup \mathcal{B}_y)$ and the $(X, \mathcal{C}_i \bigcup \mathcal{B}_i)$ are all $(n, \overrightarrow{P_3}, 2)$-ELDs, and these $n-2$ ELDs form an $(n, \overrightarrow{P_3}, 2)$-LELD. □

Theorem 2.7. *Suppose that there exists an e-FG$(3, (K_1, K_2, \dots, K_e, K_T), \sum_{i=1}^r a_i g_i)$ of type $g_1^{a_1} g_2^{a_2} \dots g_r^{a_r}$. If there is an $(mk_1 + t, \overrightarrow{P_3})$-ELPCS of type $(m^{k_1} : t)$ for each $k_1 \in K_1$, an ELF$(\overrightarrow{P_3}, (k_i + 1)\{m\})$ for each $k_i \in K_i$, $2 \le i \le e$, and an ELF$(\overrightarrow{P_3}, k\{m\})$ for each $k \in K_T$, then there is an $(m \sum_{i=1}^r a_i g_i + t + (e-1)m, \overrightarrow{P_3})$-ELPCS of type $((mg_1)^{a_1}(mg_2)^{a_2} \dots (mg_r)^{a_r} : t + (e-1)m)$.*

Proof. Let $(X, \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_e, \mathcal{A}_T)$ be an e-FG$(3, (K_1, K_2, \dots, K_e, K_T), \sum_{i=1}^r a_i g_i)$ of type $g_1^{a_1} g_2^{a_2} \dots g_r^{a_r}$. Let $S = \{\infty\} \times Z_s$, where $s = t + (e-1)m$. We shall construct the desired design on $X' = (X \times Z_m) \cup S$ with the group set $\mathcal{G}' = \{G' = G \times Z_m : G \in \mathcal{G}\}$ and the stem S, where $(X \times Z_m) \cap S = \emptyset$.

Define $G_x = \{x\} \times Z_m$ for $x \in X$ and $G_A = \{G_x : x \in A\}$ for any subset A of X. Define $S_1 = \{\infty\} \times Z_t$ and $S_i = \{\infty\} \times \{t + (i-2)m, t + (i-2)m + 1, \dots, t + (i-1)m - 1\}$ for $2 \le i \le e$.

For each block $A \in \mathcal{A}_1$, construct an $(m|A| + t, \overrightarrow{P_3})$-ELPCS of type $(m^{|A|} : t)$ on $(A \times Z_m) \cup S_1$ having group set G_A and stem S_1. Denote its path set by $\mathcal{D}_A$. By the definition, $\mathcal{D}_A$ can be partitioned into subsets $\mathcal{D}_A(x, i)$ ($(x, i) \in A \times Z_m$) and $\mathcal{D}_A(j)$ ($2 \le j \le t-1$) with the properties that each $\mathcal{D}_A(x, i)$ is the path set of an $(m|A| + t, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{m(|A|-1)}(m+t)^1$ with the long group $G_x \cup S_1$ and that each $(A \times Z_m, G_A, \mathcal{D}_A(j))$ is an $(m|A|, \overrightarrow{P_3}, 2)$-ELGDD of type $m^{|A|}$.

For each block $A \in \mathcal{A}_i$, $2 \le i \le e$, construct an ELF $(\overrightarrow{P_3}, (|A|+1)\{m\})$ on $(A \times Z_m) \cup S_i$ having group set $G_A \cup \{S_i\}$. Denote its path set by $\mathcal{C}_A$. By the definition, $\mathcal{C}_A$ can be partitioned into subsets $\mathcal{C}_A(x, i)$ ($(x, i) \in (A \times Z_m) \cup S_i$) with the property that each $\mathcal{C}_A(x, i)$ is the path set of an $(m|A|, \overrightarrow{P_3}, 2)$-ELGDD of type $m^{|A|}$ with the group set G_A when $x = \infty$ or $(G_A \cup \{S_i\}) \setminus \{G_x\}$ when $x \in A$.

For each block $A \in \mathcal{A}_T$, construct an ELF$(\overrightarrow{P_3}, |A|\{m\})$ on $A \times Z_m$ having group set G_A. Denote its path set by $\mathcal{B}_A$. By the definition, $\mathcal{B}_A$ can be partitioned into subsets $\mathcal{B}_A(x, i)$ ($(x, i) \in A \times Z_m$) with the property that each $\mathcal{B}_A(x, i)$ is the path set of an $(m(|A|-1), \overrightarrow{P_3}, 2)$-ELGDD of type $m^{|A|-1}$ with the group set $G_A \setminus \{G_x\}$.

For any $x \in X$ and $i \in Z_m$, let

$$\mathcal{F}(x, i) = \left(\bigcup_{A \in \mathcal{A}_1, x \in A} \mathcal{D}_A(x, i) \right) \bigcup \left(\bigcup_{A \in \mathcal{A}_j, 2 \le j \le e, x \in A} \mathcal{C}_A(x, i) \right) \bigcup \left(\bigcup_{A \in \mathcal{A}_T, x \in A} \mathcal{B}_A(x, i) \right).$$

For any $2 \le i \le t-1$, let

$$\mathcal{F}(\infty, i) = \bigcup_{A \in \mathcal{A}_1} \mathcal{D}_A(\infty, i).$$

For any $t+(j-2)m \le i \le t+(j-1)m-1$, $2 \le j \le e$, let

$$\mathcal{F}(\infty, i) = \bigcup_{A \in \mathcal{A}_j} \mathcal{C}_A(\infty, i).$$

Let

$$\mathcal{F} = \left(\bigcup_{x \in X, i \in Z_m} \mathcal{F}(x, i) \right) \bigcup \left(\bigcup_{2 \le i \le s-1} \mathcal{F}(\infty, i) \right).$$

For each $x \in G$ and $i \in Z_m$, $\mathcal{F}(x, i)$ is the path set of an $(m\sum_{i=1}^{r} a_i g_i + t + (e-1)m, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{m(\sum_{i=1}^{r} a_i g_i - |G|)}(m|G| + t + (e-1)m)^1$ with the long group $G' \cup S$. Each $(X', \mathcal{G}', \mathcal{F}(\infty, i))$ is an $(m\sum_{i=1}^{r} a_i g_i, \overrightarrow{P_3}, 2)$-ELGDD of type $(mg_1)^{a_1}(mg_2)^{a_2} \ldots (mg_r)^{a_r}$. So they form an $(m\sum_{i=1}^{r} a_i g_i + t + (e-1)m, \overrightarrow{P_3})$-ELPCS of type $((mg_1)^{a_1}(mg_2)^{a_2} \ldots (mg_r)^{a_r} : t + (e-1)m)$. □

3. Results for small ingredient designs

Lemma 3.1. *There does not exist a* $(3h, \overrightarrow{P_3}, 2)$*-ELGDD of type* h^3 *for any odd integer* $h > 0$.

Proof. Suppose that there exists a $(3h, \overrightarrow{P_3}, 2)$-ELGDD of type h^3; then we can construct such a design $(X, \mathcal{B})$ on Z_{3h} with group set $\{\{i, i+3, \ldots, i+3(h-1)\} : 0 \le i \le 2\}$, where $|\mathcal{B}| = 6h^2$. From the definition, we know that the three vertices in each path are from distinct groups, i.e., are distinct modulo 3. For each $B = (x, y, z) \in \mathcal{B}$, let $\hat{B} \equiv (x, y, z) \pmod 3$ be the path restricted to Z_3. Let $A = \{B \in \mathcal{B} | \hat{B} \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}\}$ and $A' = \{B \in \mathcal{B} | \hat{B} \in \{(0, 2, 1), (2, 1, 0), (1, 0, 2)\}\}$. Then it is easy to check that $\mathcal{B}$ is the disjoint union of A and A'. Since any two paths coming from A and A' respectively have no common arc, they cannot be in the same cycle in the path graph of $\mathcal{B}$. But $|A| = |A'| = 3h^2$ is odd, and neither A nor A' can be partitioned into even cycles only, which leads to a contradiction. □

By Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let* $h > 0$ *be an odd integer and* $s \ge 3$. *There does not exist an* ELF$(\overrightarrow{P_3}, 4\{h\})$ *and a* $(3h+s, \overrightarrow{P_3})$*-ELPCS of type* $(h^3 : s)$.

By an exhaustive computer search, we have:

Lemma 3.3. *There does not exist an* $(8, \overrightarrow{P_3})$*-ELPCS of type* $(2^3 : 2)$.

Now, we give direct constructions for some small ingredient designs, such as LELDs, ELFs and ELPCSs. These designs are constructed on some abelian groups by listing the corresponding initial ELD (ELGDD), which will be developed under the automorphism group to get the remaining ELDs (ELGDDs). The path sets of the initial designs are found by computer search. We may also use some additional automorphisms to shorten the path list of the initial design. The even property for each design can be checked by computer.

Lemma 3.4. *There exist both a* $(9, \overrightarrow{P_3}, 2)$*-LELD and a* $(10, \overrightarrow{P_3}, 2)$*-LELD.*

Proof. We construct the design on Z_n for each $n \in \{9, 10\}$. We list the paths of the initial $(n, \overrightarrow{P_3}, 2)$-ELD, which will be developed under the automorphism group $G = \langle (0\ 1\ 2\ \ldots\ n-3)(n-2)(n-1) \rangle$.

$n = 9$:	(0, 2, 1)	(1, 2, 0)	(2, 3, 0)	(3, 8, 0)	(4, 8, 0)	(5, 4, 0)	(6, 5, 0)	(7, 6, 0)	(8, 7, 0)
	(0, 1, 2)	(1, 4, 2)	(2, 6, 1)	(3, 7, 1)	(4, 5, 1)	(5, 7, 1)	(6, 8, 1)	(7, 5, 1)	(8, 6, 1)
	(0, 1, 3)	(1, 4, 3)	(2, 8, 3)	(3, 8, 2)	(4, 6, 2)	(5, 7, 2)	(6, 5, 2)	(7, 3, 2)	(8, 4, 2)
	(0, 2, 4)	(1, 3, 4)	(2, 7, 4)	(3, 7, 4)	(4, 6, 3)	(5, 2, 3)	(6, 4, 3)	(7, 8, 3)	(8, 6, 3)
	(0, 3, 5)	(1, 0, 5)	(2, 0, 5)	(3, 1, 5)	(4, 1, 5)	(5, 3, 4)	(6, 8, 4)	(7, 6, 4)	(8, 5, 4)
	(0, 3, 6)	(1, 0, 6)	(2, 5, 6)	(3, 0, 6)	(4, 1, 6)	(5, 3, 6)	(6, 7, 5)	(7, 3, 5)	(8, 2, 5)
	(0, 4, 7)	(1, 6, 7)	(2, 4, 7)	(3, 2, 7)	(4, 0, 7)	(5, 8, 7)	(6, 0, 7)	(7, 2, 6)	(8, 5, 6)
	(0, 4, 8)	(1, 7, 8)	(2, 1, 8)	(3, 1, 8)	(4, 5, 8)	(5, 0, 8)	(6, 2, 8)	(7, 0, 8)	(8, 1, 7)

It is readily checked that the path graph consists of a 72-cycle.

$n = 10$:

(0, 2, 1)	(1, 2, 0)	(2, 3, 0)	(3, 7, 0)	(4, 7, 0)	(5, 4, 0)	(6, 4, 0)	(7, 5, 0)	(8, 6, 0)	(9, 8, 0)
(0, 1, 2)	(1, 4, 2)	(2, 4, 1)	(3, 9, 1)	(4, 6, 1)	(5, 6, 1)	(6, 9, 1)	(7, 8, 1)	(8, 5, 1)	(9, 7, 1)
(0, 1, 3)	(1, 4, 3)	(2, 7, 3)	(3, 9, 2)	(4, 6, 2)	(5, 3, 2)	(6, 8, 2)	(7, 5, 2)	(8, 9, 2)	(9, 6, 2)
(0, 2, 4)	(1, 3, 4)	(2, 6, 4)	(3, 8, 4)	(4, 2, 3)	(5, 8, 3)	(6, 7, 3)	(7, 9, 3)	(8, 6, 3)	(9, 5, 3)
(0, 3, 5)	(1, 0, 5)	(2, 8, 5)	(3, 2, 5)	(4, 9, 5)	(5, 8, 4)	(6, 5, 4)	(7, 9, 4)	(8, 3, 4)	(9, 7, 4)
(0, 3, 6)	(1, 0, 6)	(2, 0, 6)	(3, 7, 6)	(4, 9, 6)	(5, 1, 6)	(6, 3, 5)	(7, 4, 5)	(8, 2, 5)	(9, 4, 5)
(0, 4, 7)	(1, 5, 7)	(2, 8, 7)	(3, 0, 7)	(4, 1, 7)	(5, 2, 7)	(6, 5, 7)	(7, 2, 6)	(8, 7, 6)	(9, 3, 6)
(0, 4, 8)	(1, 6, 8)	(2, 9, 8)	(3, 1, 8)	(4, 3, 8)	(5, 0, 8)	(6, 7, 8)	(7, 1, 8)	(8, 1, 7)	(9, 0, 7)
(0, 5, 9)	(1, 5, 9)	(2, 1, 9)	(3, 1, 9)	(4, 8, 9)	(5, 6, 9)	(6, 0, 9)	(7, 2, 9)	(8, 0, 9)	(9, 0, 8)

It is readily checked that the path graph consists of a 28-cycle and a 62-cycle. □

Lemma 3.5. *There exists an* ELF($\overrightarrow{P_3}$, 4{2}).

Proof. We construct the design on Z_8 with group set $\{\{i, i+4\} : 0 \le i \le 3\}$. We first construct below an initial (6, $\overrightarrow{P_3}$, 2)-ELGDD of type 2^3 on the group set $\{\{i, i+4\} : 1 \le i \le 3\}$ with the path graph consisting of four 6-cycles.

(2, 1, 7)	(2, 3, 1)	(2, 5, 3)	(2, 7, 5)	(3, 1, 6)	(3, 2, 1)	(3, 5, 2)	(3, 6, 5)
(6, 3, 5)	(6, 5, 7)	(1, 2, 3)	(6, 7, 1)	(7, 2, 5)	(7, 5, 6)	(7, 6, 1)	(1, 6, 7)
(5, 2, 7)	(7, 1, 2)	(5, 6, 3)	(1, 7, 2)	(5, 3, 2)	(5, 7, 6)	(6, 1, 3)	(1, 3, 6)

Developing the above paths under the automorphism group $G = \langle \pi : i \to i+1 \rangle$, we get eight (6, $\overrightarrow{P_3}$, 2)-ELGDDs altogether, which form an ELF($\overrightarrow{P_3}$, 4{2}). □

Lemma 3.6. *There exists an* ELF($\overrightarrow{P_3}$, 5{4}).

Proof. We construct the design on Z_{20} with group set $\{\{i, i+5, i+10, i+15\} : 0 \le i \le 4\}$. We list the path set of an initial (16, $\overrightarrow{P_3}$, 2)-ELGDD of type 4^4 on the group set $\{\{i, i+5, i+10, i+15\} : 1 \le i \le 4\}$ with a multiplicative automorphism group $G' = \langle (0)(1\ 3\ 9\ 7)(2\ 6\ 18\ 14)(4\ 12\ 16\ 8)(5\ 15)(10)(11\ 13\ 19\ 17) \rangle$.

(19, 12, 8)	(13, 2, 6)	(9, 16, 17)	(9, 8, 7)	(6, 19, 8)	(6, 13, 7)	(1, 14, 2)
(13, 7, 14)	(16, 18, 4)	(16, 7, 9)	(1, 19, 17)	(4, 12, 11)	(16, 3, 7)	(14, 17, 1)
(13, 6, 4)	(6, 4, 17)	(19, 8, 1)	(13, 6, 17)	(17, 6, 9)	(6, 3, 12)	(11, 19, 13)
(6, 8, 2)	(6, 19, 13)	(8, 17, 11)	(1, 17, 8)	(18, 2, 9)	(17, 16, 8)	(9, 17, 8)
(8, 6, 14)	(7, 1, 13)	(4, 18, 7)	(14, 8, 16)	(18, 1, 14)	(8, 2, 19)	(1, 8, 14)
(4, 11, 18)	(1, 7, 4)	(19, 3, 6)	(8, 16, 4)	(11, 3, 17)	(7, 3, 9)	(14, 8, 11)
(12, 3, 4)	(7, 19, 6)	(2, 14, 18)	(1, 18, 9)	(4, 7, 6)	(19, 11, 7)	

The path graph of this ELGDD consists of one 188-cycle and one 4-cycle. Developing the initial ELGDD under the automorphism group $G = \langle \pi : i \to i+1 \rangle$, we get twenty (16, $\overrightarrow{P_3}$, 2)-ELGDDs altogether, which form an ELF($\overrightarrow{P_3}$, 5{4}). □

Lemma 3.7. *There exists an* ELF($\overrightarrow{P_3}$, 6{4}).

Proof. We construct the design on Z_{24} with group set $\{\{i, i+6, i+12, i+18\} : 0 \le i \le 5\}$. We list the path set of an initial (20, $\overrightarrow{P_3}$, 2)-ELGDD of type 4^5 on the group set $\{\{i, i+6, i+12, i+18\} : 1 \le i \le 5\}$ with a multiplicative automorphism group $G' = \langle \eta : i \to 17i \rangle$.

(15, 5, 20)	(19, 3, 22)	(11, 13, 21)	(11, 7, 20)	(14, 3, 7)	(9, 14, 22)	(10, 17, 19)
(9, 14, 7)	(5, 19, 14)	(15, 4, 19)	(8, 16, 21)	(15, 11, 16)	(9, 23, 19)	(7, 21, 22)
(11, 19, 22)	(2, 3, 16)	(3, 16, 23)	(9, 4, 13)	(21, 5, 22)	(21, 2, 13)	(21, 19, 20)
(14, 17, 22)	(19, 17, 15)	(13, 15, 16)	(3, 19, 14)	(20, 3, 22)	(20, 10, 15)	(15, 20, 22)
(16, 15, 23)	(11, 20, 1)	(11, 21, 10)	(20, 23, 16)	(14, 5, 16)	(10, 14, 21)	(15, 14, 17)
(8, 19, 23)	(7, 3, 17)	(10, 13, 23)	(13, 21, 20)	(7, 16, 8)	(23, 22, 20)	(3, 19, 16)
(8, 1, 4)	(9, 11, 8)	(1, 16, 20)	(9, 13, 17)	(16, 14, 19)	(7, 23, 21)	(10, 19, 9)
(1, 5, 8)	(14, 9, 10)	(17, 20, 7)	(17, 2, 13)	(20, 19, 11)	(20, 9, 7)	(15, 16, 2)
(11, 13, 4)	(21, 8, 23)	(3, 13, 20)	(20, 11, 3)	(10, 23, 7)	(16, 17, 13)	(22, 17, 3)
(21, 7, 10)	(14, 16, 1)	(1, 20, 15)	(8, 19, 15)	(8, 22, 19)	(21, 1, 16)	(19, 4, 11)
(21, 23, 1)	(4, 1, 21)	(13, 22, 2)	(13, 14, 23)	(23, 15, 19)	(16, 2, 3)	(11, 9, 16)
(11, 16, 13)	(15, 5, 13)	(14, 3, 23)	(16, 19, 2)	(7, 22, 20)	(17, 21, 22)	(10, 23, 13)
(2, 21, 4)	(23, 2, 10)	(9, 1, 10)	(11, 21, 8)	(4, 7, 11)	(1, 9, 16)	(16, 3, 17)
(13, 11, 14)	(17, 14, 19)	(1, 8, 4)	(10, 9, 5)	(1, 23, 22)	(3, 2, 11)	(8, 9, 17)
(13, 10, 8)	(7, 22, 15)	(15, 17, 7)	(3, 10, 1)	(16, 7, 9)	(11, 4, 3)	(21, 17, 19)

(3, 4, 5)	(22, 23, 20)	(10, 7, 17)	(7, 5, 14)	(1, 8, 21)	(4, 1, 2)	(4, 14, 13)
(14, 16, 5)	(22, 7, 9)	(1, 17, 3)	(22, 2, 21)	(14, 9, 11)	(13, 20, 21)	(22, 14, 5)
(19, 10, 9)	(4, 20, 1)	(19, 2, 23)	(1, 20, 9)	(19, 15, 10)	(3, 4, 2)	(1, 15, 2)
(2, 4, 15)	(22, 21, 11)	(10, 11, 2)	(4, 8, 17)	(8, 13, 16)	(23, 10, 8)	(13, 3, 5)
(10, 13, 3)	(22, 15, 1)	(8, 15, 22)	(10, 14, 11)	(9, 20, 4)	(14, 13, 15)	(5, 10, 20)
(7, 16, 9)	(8, 5, 13)	(20, 21, 13)	(5, 7, 3)	(23, 10, 2)	(13, 2, 15)	(7, 20, 5)
(17, 14, 1)	(10, 11, 1)	(5, 16, 19)	(2, 17, 22)	(5, 2, 1)	(7, 8, 3)	(4, 23, 20)
(23, 15, 7)	(4, 5, 9)	(1, 17, 10)	(5, 1, 9)	(5, 9, 2)	(20, 16, 23)	

The path graph of this ELGDD consists of two 152-cycles, two 6-cycles and one 4-cycle. Developing the initial ELGDD under the automorphism group $G = \langle \pi : i \to i + 1 \rangle$, we get twenty four $(20, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form an ELF$(\overrightarrow{P_3}, 6\{4\})$. □

Lemma 3.8. *There exists an* $(11, \overrightarrow{P_3})$*-ELPCS of type* $(3^3 : 2)$.

Proof. We construct the design on Z_{11} with the group set $\{\{i, i+3, i+6\} : 0 \le i \le 2\}$ and the stem $\{9, 10\}$. We first construct an initial $(11, \overrightarrow{P_3}, 2)$-ELGDD of type $1^6 5^1$ with the long group $\{0, 3, 6, 9, 10\}$ and the following path set:

(9, 2, 4)	(4, 3, 2)	(0, 2, 8)	(5, 9, 4)	(0, 5, 2)	(0, 8, 5)	(1, 2, 0)	(1, 4, 3)	(3, 1, 4)
(7, 9, 2)	(2, 8, 3)	(10, 7, 2)	(7, 8, 10)	(7, 0, 1)	(7, 0, 4)	(2, 1, 6)	(5, 6, 7)	(7, 4, 6)
(2, 6, 5)	(10, 2, 7)	(4, 9, 5)	(1, 3, 5)	(8, 4, 6)	(1, 0, 7)	(8, 4, 9)	(6, 7, 4)	(5, 1, 10)
(0, 5, 4)	(7, 10, 5)	(5, 7, 9)	(6, 4, 1)	(6, 2, 5)	(4, 1, 6)	(10, 4, 5)	(6, 8, 2)	(4, 0, 7)
(2, 5, 0)	(9, 8, 7)	(2, 9, 7)	(2, 3, 8)	(8, 7, 10)	(8, 10, 7)	(6, 5, 8)	(7, 3, 8)	(4, 7, 3)
(5, 8, 0)	(3, 7, 5)	(5, 3, 2)	(8, 5, 3)	(0, 2, 7)	(4, 0, 1)	(9, 5, 1)	(2, 4, 10)	(3, 4, 8)
(3, 1, 2)	(1, 0, 4)	(1, 8, 9)	(1, 5, 10)	(3, 7, 1)	(8, 2, 0)	(5, 10, 1)	(10, 1, 8)	(7, 5, 0)
(1, 9, 8)	(1, 10, 2)	(7, 1, 3)	(4, 10, 8)	(8, 9, 1)	(3, 4, 7)	(8, 3, 5)	(2, 1, 9)	(10, 8, 1)
(8, 6, 4)	(6, 1, 7)	(2, 6, 1)	(0, 8, 1)	(4, 5, 9)	(2, 10, 4)	(5, 2, 3)	(5, 7, 6)	(8, 6, 2)
(7, 2, 9)	(9, 7, 8)	(1, 7, 6)	(4, 8, 0)	(5, 6, 8)	(9, 1, 5)	(9, 4, 2)	(10, 5, 4)	(4, 2, 10)

It is readily checked that the path graph consists of four 6-cycles and one 66-cycle. Developing the paths under the automorphism group $G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9)(10) \rangle$, we get nine $(11, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form an $(11, \overrightarrow{P_3})$-ELPCS of type $(3^3 : 2)$. □

Lemma 3.9. *There exists a* $(14, \overrightarrow{P_3})$*-ELPCS of type* $(4^3 : 2)$.

Proof. We construct the design on Z_{14} with the group set $\{\{i, i+3, i+6, i+9\} : 0 \le i \le 2\}$ and the stem $\{12, 13\}$. We list below the path sets of two initial $(14, \overrightarrow{P_3}, 2)$-ELGDDs of type $1^8 6^1$ with the long group $\{0, 3, 6, 9, 12, 13\}$, both of which have an automorphism group $G' = \langle (0)(1\ 5)(2\ 10)(3)(4\ 8)(6)(7\ 11)(9)(12)(13) \rangle$.

The first initial ELGDD with the path graph consisting of two 74-cycles and one 4-cycle:

(1, 13, 8)	(0, 2, 10)	(8, 0, 4)	(8, 11, 9)	(5, 12, 7)	(6, 1, 7)	(10, 9, 8)
(2, 4, 10)	(7, 1, 0)	(11, 10, 13)	(13, 5, 1)	(9, 2, 1)	(3, 11, 8)	(12, 2, 7)
(3, 4, 10)	(9, 4, 11)	(11, 13, 7)	(7, 12, 2)	(1, 11, 13)	(2, 0, 5)	(4, 2, 3)
(0, 4, 1)	(5, 3, 8)	(7, 1, 9)	(1, 6, 7)	(11, 12, 4)	(13, 10, 2)	(5, 7, 6)
(11, 1, 8)	(7, 5, 3)	(1, 10, 3)	(12, 4, 8)	(8, 11, 0)	(4, 13, 2)	(2, 13, 1)
(7, 3, 10)	(5, 10, 0)	(4, 8, 6)	(5, 4, 12)	(8, 1, 13)	(13, 4, 11)	(8, 0, 11)
(0, 5, 11)	(5, 8, 1)	(5, 2, 9)	(2, 7, 11)	(2, 8, 6)	(5, 0, 2)	(3, 10, 1)
(9, 11, 2)	(7, 8, 12)	(2, 6, 8)	(11, 10, 6)	(4, 9, 11)	(10, 8, 13)	(2, 12, 7)
(6, 10, 4)	(1, 6, 2)	(11, 9, 1)	(2, 5, 9)	(3, 11, 7)	(12, 1, 5)	(8, 3, 1)
(8, 9, 2)	(6, 7, 10)	(2, 11, 0)	(4, 2, 12)	(8, 3, 5)	(13, 11, 4)	(12, 1, 2)
(0, 11, 8)	(7, 6, 1)	(6, 8, 5)	(10, 5, 12)	(9, 5, 8)	(2, 11, 3)	

The second initial ELGDD with the path graph consisting of two 76-cycles:

(7, 11, 13)	(9, 4, 1)	(10, 2, 12)	(4, 8, 1)	(10, 12, 8)	(1, 8, 9)	(8, 12, 7)
(4, 7, 6)	(8, 3, 2)	(6, 7, 2)	(11, 5, 3)	(1, 9, 7)	(10, 13, 2)	(4, 9, 2)
(11, 2, 9)	(11, 7, 12)	(9, 1, 2)	(0, 5, 2)	(7, 13, 8)	(5, 12, 1)	(11, 2, 6)
(6, 5, 8)	(8, 4, 13)	(11, 9, 5)	(1, 6, 10)	(5, 0, 10)	(7, 4, 0)	(9, 2, 11)
(11, 1, 7)	(3, 10, 4)	(4, 3, 7)	(8, 1, 12)	(12, 11, 10)	(10, 4, 0)	(2, 3, 5)
(8, 13, 1)	(7, 3, 4)	(6, 1, 5)	(11, 0, 1)	(2, 10, 1)	(3, 5, 2)	(0, 4, 11)
(0, 7, 8)	(1, 4, 6)	(11, 6, 2)	(12, 10, 5)	(4, 10, 3)	(6, 4, 11)	(10, 6, 11)
(8, 5, 0)	(3, 4, 7)	(12, 2, 4)	(13, 7, 5)	(5, 10, 13)	(10, 5, 13)	(5, 7, 12)
(9, 11, 4)	(4, 12, 8)	(5, 11, 0)	(13, 4, 2)	(12, 5, 7)	(3, 11, 5)	(13, 5, 4)
(5, 1, 3)	(7, 4, 2)	(1, 13, 11)	(1, 9, 8)	(0, 2, 5)	(5, 6, 8)	(10, 0, 4)
(8, 2, 9)	(10, 0, 7)	(10, 7, 9)	(2, 4, 6)	(2, 7, 3)	(13, 2, 7)	

Let $G = \langle (0\ 2\ 4\ 6\ 8\ 10)(1\ 3\ 5\ 7\ 9\ 11)(12)(13) \rangle$. Developing the above two initial designs under the automorphism group G, we get twelve $(14, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form a $(14, \overrightarrow{P_3})$-ELPCS of type $(4^3 : 2)$. □

Lemma 3.10. *There exists an* $(18, \overrightarrow{P_3})$*-ELPCS of type* $(4^4 : 2)$.

Proof. We construct the design on Z_{18} with the group set $\{\{i, i+4, i+8, i+12\} : 0 \le i \le 3\}$ and the stem $\{16, 17\}$. Let

$$G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)(16)(17) \rangle, \quad \text{and} \quad G' = \langle \eta : i \to 7i \rangle.$$

We list below the path set of an initial $(18, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}6^1$ on Z_{18} with the long group $\{0, 4, 8, 12, 16, 17\}$ and the automorphism group G', where the path graph consists of one 264-cycle and three 4-cycles.

(6, 12, 1)	(1, 5, 3)	(6, 4, 14)	(1, 13, 14)	(5, 14, 17)	(8, 14, 6)	(0, 10, 6)
(6, 15, 2)	(8, 5, 13)	(15, 6, 3)	(3, 2, 8)	(4, 15, 2)	(13, 11, 12)	(9, 16, 10)
(15, 10, 1)	(2, 11, 6)	(13, 4, 9)	(15, 0, 13)	(10, 2, 12)	(15, 9, 8)	(4, 2, 15)
(8, 15, 3)	(0, 1, 3)	(11, 17, 1)	(4, 2, 9)	(7, 3, 14)	(13, 3, 10)	(4, 9, 3)
(5, 17, 6)	(3, 0, 14)	(16, 2, 15)	(2, 17, 13)	(7, 10, 8)	(4, 11, 13)	(17, 5, 7)
(17, 2, 3)	(0, 3, 13)	(13, 7, 16)	(5, 13, 14)	(3, 4, 11)	(6, 13, 7)	(10, 17, 15)
(7, 13, 12)	(10, 3, 9)	(14, 0, 3)	(8, 6, 1)	(6, 0, 13)	(12, 9, 13)	(3, 5, 1)
(2, 8, 9)	(1, 14, 16)	(3, 16, 13)	(17, 3, 13)	(2, 1, 10)	(10, 1, 8)	(11, 4, 5)
(13, 8, 1)	(9, 2, 16)	(4, 3, 6)	(3, 1, 16)	(9, 17, 2)	(1, 17, 15)	(9, 16, 14)
(13, 0, 2)	(16, 6, 11)	(5, 8, 3)	(4, 13, 1)	(8, 6, 14)	(16, 1, 2)	(1, 12, 6)
(11, 3, 0)	(9, 11, 15)	(9, 5, 12)	(5, 1, 0)	(3, 9, 15)	(6, 2, 0)	(0, 15, 14)
(6, 3, 17)	(16, 7, 6)	(9, 0, 6)	(13, 2, 6)	(4, 10, 7)	(0, 15, 1)	(14, 11, 7)
(4, 6, 5)	(17, 13, 6)	(6, 16, 5)	(14, 2, 13)	(1, 3, 12)	(11, 10, 8)	(1, 8, 11)
(5, 6, 10)	(14, 10, 0)	(6, 11, 16)	(14, 1, 17)	(2, 9, 8)	(13, 2, 5)	(2, 4, 14)
(9, 4, 3)	(14, 3, 12)	(7, 11, 0)	(15, 6, 17)	(10, 4, 6)	(6, 2, 12)	(17, 6, 9)
(14, 5, 16)	(11, 9, 12)	(13, 15, 17)	(2, 10, 12)	(14, 13, 5)	(7, 1, 6)	(16, 9, 3)
(15, 10, 11)	(5, 2, 4)	(1, 11, 17)	(2, 14, 7)	(9, 1, 0)	(3, 15, 4)	(5, 8, 1)
(12, 11, 6)	(1, 12, 7)	(10, 16, 5)	(13, 8, 14)	(6, 15, 11)	(17, 1, 14)	(8, 13, 9)
(0, 7, 15)	(1, 7, 9)	(4, 7, 14)	(1, 4, 5)	(9, 1, 4)	(3, 6, 9)	(9, 4, 1)
(14, 1, 9)	(11, 16, 13)	(11, 5, 9)	(16, 9, 7)	(1, 15, 13)		

Developing the above initial design under the automorphism group G, we get sixteen $(18, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form an $(18, \overrightarrow{P_3})$-ELPCS of type $(4^4 : 2)$. □

Lemma 3.11. *There exists a* $(22, \overrightarrow{P_3})$*-ELPCS of type* $(4^5 : 2)$.

Proof. We construct the design on Z_{22} with the group set $\{\{i, i+5, i+10, i+15\} : 0 \le i \le 4\}$ and the stem $\{20, 21\}$. Let

$$G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19)(20)(21) \rangle, \quad \text{and}$$
$$G' = \langle (0)(1\ 3\ 9\ 7)(2\ 6\ 18\ 14)(4\ 12\ 16\ 8)(5\ 15)(10)(11\ 13\ 19\ 17)(20)(21) \rangle.$$

We list below the path set of an initial $(22, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{16}6^1$ on Z_{22} with the long group $\{0, 5, 10, 15, 20, 21\}$ and the automorphism group G', where the path graph consists of one 180-cycle, one 88-cycle, two 60-cycles, one 40-cycle and one 4-cycle.

(5, 11, 12)	(6, 10, 13)	(1, 14, 16)	(0, 3, 13)	(1, 16, 12)	(7, 0, 19)	(7, 8, 17)
(3, 18, 17)	(18, 8, 0)	(6, 0, 4)	(18, 17, 4)	(19, 12, 5)	(0, 16, 7)	(19, 20, 12)
(7, 21, 14)	(10, 9, 4)	(2, 1, 5)	(8, 7, 17)	(5, 1, 3)	(12, 7, 14)	(7, 10, 2)
(7, 6, 0)	(15, 11, 6)	(7, 3, 6)	(19, 21, 1)	(18, 5, 16)	(21, 2, 3)	(10, 12, 9)
(21, 13, 17)	(9, 3, 1)	(6, 7, 2)	(4, 10, 16)	(10, 3, 19)	(12, 8, 18)	(1, 19, 20)
(11, 0, 13)	(8, 20, 14)	(14, 17, 19)	(4, 21, 12)	(5, 6, 11)	(0, 6, 4)	(16, 6, 5)
(6, 9, 17)	(9, 6, 21)	(13, 9, 21)	(3, 5, 4)	(18, 21, 9)	(12, 7, 4)	(17, 2, 4)
(17, 9, 2)	(16, 14, 9)	(20, 8, 11)	(11, 8, 4)	(19, 16, 2)	(1, 8, 4)	(5, 3, 4)
(11, 16, 0)	(4, 10, 19)	(15, 4, 2)	(16, 18, 3)	(19, 11, 10)	(20, 13, 7)	(8, 6, 3)
(17, 7, 5)	(4, 8, 13)	(4, 15, 14)	(17, 12, 1)	(11, 15, 19)	(7, 8, 15)	(9, 13, 15)
(8, 13, 19)	(16, 3, 0)	(1, 5, 14)	(8, 7, 10)	(10, 2, 6)	(7, 3, 9)	(8, 12, 20)
(21, 13, 12)	(17, 0, 9)	(15, 6, 11)	(2, 20, 9)	(18, 1, 20)	(18, 8, 15)	(11, 13, 14)
(19, 15, 4)	(7, 20, 11)	(11, 17, 1)	(18, 11, 2)	(14, 18, 10)	(4, 13, 15)	(7, 19, 10)
(6, 17, 18)	(2, 14, 12)	(20, 1, 12)	(12, 11, 9)	(14, 15, 1)	(0, 14, 2)	(15, 13, 3)
(7, 13, 1)	(11, 18, 2)	(18, 15, 3)	(20, 6, 14)	(21, 4, 6)	(17, 14, 20)	(18, 16, 21)
(2, 16, 13)	(4, 11, 21)	(17, 18, 19)				

Developing the above initial design under the automorphism group G, we get twenty $(22, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form a $(22, \overrightarrow{P_3})$-ELPCS of type $(4^5 : 2)$. □

Lemma 3.12. *There exists a* $(20, \overrightarrow{P_3})$*-ELPCS of type* $(6^3 : 2)$.

Proof. We construct the design on Z_{20} with the group set $\{\{i, i+3, \ldots, i+15\} : 0 \le i \le 2\}$ and the stem $\{18, 19\}$. Let

$$G = \langle (0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16)(1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17)(18)(19), (0)(1\ 5\ 7\ 17\ 13\ 11)(2\ 10\ 14\ 16\ 8\ 4)(3\ 15)(6\ 12)(9)(18)(19)\rangle, \quad \text{and}$$

$$G' = \langle (0)(1\ 7\ 13)(2\ 14\ 8)(3)(4\ 10\ 16)(5\ 17\ 11)(6)(9)(12)(15)(18)(19)\rangle.$$

We list below the path set of a $(20, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}8^1$ on Z_{20} with the long group $\{0, 3, 6, 9, 12, 15, 18, 19\}$ and the automorphism group G'. The path graph consists of one 54-cycle, one 264-cycle and one 6-cycle.

(7, 2, 19)	(10, 3, 16)	(2, 6, 14)	(4, 16, 15)	(3, 13, 1)	(17, 13, 19)	(7, 17, 11)
(8, 0, 5)	(5, 0, 17)	(15, 11, 16)	(11, 12, 14)	(3, 4, 8)	(8, 4, 17)	(3, 8, 5)
(4, 0, 16)	(0, 8, 10)	(17, 18, 4)	(4, 12, 17)	(4, 10, 0)	(0, 2, 5)	(4, 19, 11)
(1, 7, 0)	(1, 5, 2)	(15, 16, 13)	(5, 14, 10)	(13, 5, 18)	(16, 10, 12)	(7, 15, 13)
(4, 1, 3)	(6, 17, 7)	(6, 8, 2)	(13, 7, 3)	(4, 18, 8)	(5, 16, 18)	(2, 6, 4)
(8, 11, 0)	(12, 4, 7)	(14, 12, 7)	(7, 12, 8)	(11, 9, 17)	(9, 14, 2)	(0, 1, 14)
(1, 16, 11)	(17, 5, 6)	(13, 4, 9)	(8, 1, 19)	(13, 18, 11)	(11, 19, 7)	(2, 7, 18)
(17, 6, 7)	(1, 2, 8)	(3, 7, 10)	(19, 17, 16)	(17, 11, 3)	(9, 7, 5)	(2, 11, 15)
(12, 13, 4)	(16, 19, 2)	(7, 6, 10)	(2, 11, 12)	(14, 1, 13)	(18, 1, 2)	(17, 2, 0)
(14, 15, 16)	(16, 8, 19)	(2, 10, 16)	(0, 16, 7)	(10, 5, 9)	(17, 19, 10)	(14, 11, 3)
(5, 1, 12)	(18, 17, 1)	(7, 9, 16)	(15, 13, 5)	(9, 2, 13)	(13, 14, 12)	(6, 1, 16)
(2, 3, 8)	(13, 0, 7)	(12, 17, 14)	(10, 2, 18)	(19, 1, 11)	(10, 3, 17)	(1, 14, 15)
(9, 11, 16)	(5, 13, 15)	(4, 9, 13)	(10, 8, 13)	(10, 15, 14)	(6, 11, 17)	(16, 17, 13)
(2, 18, 13)	(19, 4, 14)	(5, 15, 2)	(11, 14, 9)	(11, 10, 1)	(13, 16, 6)	(14, 3, 17)
(18, 14, 4)	(11, 4, 2)	(4, 7, 6)	(14, 8, 9)	(8, 16, 6)	(18, 4, 17)	(1, 9, 16)
(12, 4, 5)	(19, 2, 1)	(15, 5, 14)				

Developing the above initial design under the automorphism group G, we get eighteen $(20, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form a $(20, \overrightarrow{P_3})$-ELPCS of type $(6^3 : 2)$. □

Lemma 3.13. *There exists a* $(32, \overrightarrow{P_3})$*-ELPCS of type* $(6^5 : 2)$.

Proof. We construct the design on Z_{32} with the group set $\{\{i, i+5, \ldots, i+25\} : 0 \le i \le 4\}$ and the stem $\{30, 31\}$. Let

$$G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29)(30)(31)\rangle, \quad \text{and}$$

$$G' = \langle (0)(1\ 7\ 19\ 13)(2\ 14\ 8\ 26)(3\ 21\ 27\ 9)(4\ 28\ 16\ 22)(5)(6\ 12\ 24\ 18)(10)(11\ 17\ 29\ 23)(15)(20)(25)(30)(31)\rangle.$$

We list below the path set of a $(32, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{24}8^1$ on Z_{32} with the long group $\{0, 5, 10, 15, 20, 25, 30, 31\}$ and the automorphism group G'. The path graph consists of one 752-cycle, four 36-cycles, two 18-cycles and one 4-cycle.

(24, 30, 28)	(28, 17, 14)	(19, 24, 25)	(12, 7, 13)	(24, 8, 10)	(15, 13, 23)	(29, 17, 12)
(26, 31, 29)	(27, 8, 2)	(2, 13, 27)	(6, 13, 25)	(16, 5, 7)	(19, 4, 6)	(11, 23, 26)
(28, 30, 11)	(16, 9, 27)	(22, 17, 13)	(8, 22, 20)	(31, 21, 13)	(17, 29, 11)	(21, 23, 30)
(9, 29, 10)	(12, 2, 18)	(18, 7, 23)	(16, 26, 12)	(10, 22, 26)	(27, 18, 4)	(23, 28, 16)
(28, 30, 16)	(5, 13, 23)	(25, 3, 7)	(23, 22, 25)	(0, 22, 12)	(28, 29, 4)	(23, 13, 7)
(2, 24, 15)	(18, 11, 28)	(18, 14, 9)	(5, 28, 18)	(17, 12, 19)	(7, 10, 19)	(15, 17, 8)
(6, 4, 27)	(16, 24, 21)	(6, 25, 29)	(21, 30, 8)	(20, 7, 2)	(16, 15, 24)	(2, 4, 1)
(5, 8, 9)	(12, 22, 1)	(3, 16, 30)	(21, 16, 25)	(15, 16, 9)	(24, 17, 2)	(26, 9, 19)
(12, 21, 15)	(26, 25, 11)	(18, 1, 24)	(19, 8, 26)	(24, 5, 21)	(8, 3, 26)	(22, 19, 2)
(20, 8, 11)	(17, 20, 16)	(14, 26, 5)	(10, 13, 19)	(15, 23, 13)	(0, 14, 7)	(9, 13, 24)
(19, 23, 1)	(3, 19, 4)	(21, 14, 24)	(20, 2, 12)	(13, 30, 22)	(28, 26, 20)	(31, 22, 9)
(14, 29, 0)	(3, 16, 19)	(12, 20, 27)	(31, 17, 18)	(15, 24, 16)	(22, 28, 11)	(22, 9, 10)
(29, 7, 31)	(20, 12, 1)	(0, 6, 11)	(1, 25, 8)	(4, 8, 11)	(3, 5, 29)	(19, 0, 28)
(16, 7, 1)	(30, 6, 23)	(17, 30, 1)	(7, 14, 22)	(29, 22, 16)	(25, 28, 14)	(15, 21, 6)
(2, 20, 18)	(17, 3, 15)	(2, 5, 28)	(7, 5, 17)	(11, 15, 2)	(26, 31, 12)	(4, 30, 27)
(3, 11, 26)	(13, 31, 27)	(11, 12, 6)	(28, 13, 22)	(23, 24, 14)	(11, 21, 10)	(29, 23, 9)
(18, 27, 2)	(12, 22, 16)	(17, 19, 20)	(11, 3, 30)	(13, 2, 17)	(5, 12, 14)	(28, 15, 2)
(1, 22, 0)	(2, 19, 22)	(21, 4, 20)	(8, 15, 1)	(4, 7, 15)	(11, 22, 0)	(5, 8, 28)
(3, 17, 13)	(2, 14, 25)	(17, 3, 8)	(0, 2, 21)	(8, 0, 27)	(30, 18, 12)	(27, 25, 3)

(22, 24, 3)	(18, 10, 8)	(24, 28, 23)	(10, 18, 23)	(29, 26, 19)	(19, 31, 8)	(21, 6, 0)
(5, 6, 1)	(7, 10, 29)	(19, 18, 30)	(4, 26, 30)	(4, 9, 8)	(1, 23, 15)	(25, 12, 23)
(16, 23, 0)	(0, 21, 16)	(17, 23, 24)	(12, 24, 30)	(27, 23, 19)	(8, 1, 14)	(4, 27, 31)
(19, 0, 7)	(23, 25, 12)	(23, 5, 21)	(26, 29, 10)	(20, 21, 28)	(3, 29, 5)	(27, 26, 23)
(31, 1, 17)	(23, 22, 27)	(12, 4, 5)	(1, 26, 18)	(29, 3, 27)	(8, 15, 16)	(3, 27, 24)
(3, 12, 15)	(8, 17, 31)	(9, 12, 27)	(30, 7, 4)	(11, 8, 16)	(30, 12, 8)	(27, 13, 22)
(31, 13, 1)	(24, 3, 13)	(11, 24, 5)	(19, 3, 22)	(14, 2, 26)	(18, 9, 13)	(19, 27, 9)
(31, 11, 8)	(1, 21, 12)	(26, 9, 3)	(25, 16, 4)	(0, 6, 14)	(4, 19, 5)	(21, 25, 14)
(24, 2, 0)	(25, 1, 12)	(14, 3, 24)	(9, 29, 18)	(7, 18, 10)	(1, 9, 5)	(13, 1, 18)
(23, 3, 11)	(30, 8, 21)	(1, 8, 31)	(3, 20, 22)	(25, 13, 3)	(10, 4, 18)	(7, 12, 20)
(18, 24, 6)	(10, 21, 3)	(2, 10, 9)	(24, 23, 16)	(30, 26, 28)	(8, 12, 13)	(14, 19, 17)
(13, 15, 21)	(24, 22, 8)	(10, 2, 28)	(2, 22, 6)	(7, 20, 11)	(27, 16, 17)	(19, 11, 27)
(27, 30, 9)	(3, 0, 11)	(8, 24, 11)	(4, 10, 29)	(18, 0, 29)	(12, 24, 31)	(22, 2, 1)
(20, 1, 3)	(22, 23, 25)	(8, 29, 30)	(7, 9, 2)	(9, 0, 1)	(11, 20, 29)	(26, 1, 28)
(16, 10, 18)	(4, 28, 24)	(18, 3, 20)				

Developing the above initial design under the automorphism group G, we get thirty $(32, \overrightarrow{P_3}, 2)$-ELGDDs altogether, which form a $(32, \overrightarrow{P_3})$-ELPCS of type $(6^5 : 2)$. □

Lemma 3.14. *There exists a* $(23, \overrightarrow{P_3})$*-ELPCS of type* $(6^3 : 5)$.

Proof. We construct the design on Z_{23} with group set $\{\{i, i+3, \ldots, i+15\} : 0 \le i \le 2\}$ and stem $\{18, 19, 20, 21, 22\}$. Let

$$G_1 = \langle (0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16)(1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17)(18)(19)(20)(21)(22) \rangle,$$

$$G_2 = \langle (0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16)(1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17)(18)(19)(20)(21)(22), \\ (0)(1\ 5\ 7\ 17\ 13\ 11)(2\ 10\ 14\ 16\ 8\ 4)(3\ 15)(6\ 12)(9)(18)(19)(20)(21)(22) \rangle,$$

$$G' = \langle (0)(1\ 7\ 13)(2\ 14\ 8)(3)(4\ 10\ 16)(5\ 17\ 11)(6)(9)(12)(15)(18)(19)(20)(21)(22) \rangle, \quad \text{and}$$

$$G'' = \langle (0)(1\ 5\ 7\ 17\ 13\ 11)(2\ 10\ 14\ 16\ 8\ 4)(3\ 15)(6\ 12)(9)(18)(19)(20)(21)(22), \\ (0\ 6\ 12)(1\ 7\ 13)(2\ 8\ 14)(3\ 9\ 15)(4\ 10\ 16)(5\ 11\ 17)(18)(19)(20)(21)(22) \rangle.$$

We list below the path set of an $(18, \overrightarrow{P_3}, 2)$-ELGDD of type 6^3 on Z_{18} with group set $\{\{i, i+3, \ldots, i+15\} : 0 \le i \le 2\}$ and the automorphism group G''. The path graph consists of six 36-cycles.

(4, 9, 5)	(17, 16, 3)	(10, 8, 6)	(9, 10, 17)	(9, 8, 16)	(0, 5, 16)	(11, 3, 13)
(6, 1, 17)	(13, 17, 0)	(14, 0, 16)	(5, 6, 10)	(14, 7, 3)		

Then, we list below the path set of a $(23, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}11^1$ on Z_{23} with the long group {0, 3, 6, 9, 12, 15, 18, 19, 20, 21, 22} and the automorphism group G'. The path graph consists of one 138-cycle, three 84-cycles and one 6-cycle.

(21, 7, 8)	(6, 7, 1)	(16, 22, 11)	(19, 8, 10)	(1, 7, 12)	(10, 16, 6)	(13, 5, 22)
(16, 3, 7)	(7, 4, 9)	(19, 1, 11)	(11, 10, 22)	(20, 16, 17)	(19, 7, 2)	(2, 6, 11)
(7, 12, 17)	(18, 4, 2)	(14, 7, 22)	(6, 17, 8)	(10, 21, 5)	(18, 10, 17)	(14, 22, 10)
(2, 0, 5)	(14, 9, 2)	(4, 14, 22)	(15, 11, 17)	(20, 5, 4)	(4, 12, 16)	(5, 16, 12)
(8, 15, 10)	(1, 10, 0)	(16, 21, 2)	(10, 20, 2)	(17, 20, 1)	(22, 2, 1)	(8, 1, 18)
(16, 19, 5)	(16, 17, 20)	(2, 12, 13)	(4, 16, 3)	(4, 17, 12)	(17, 10, 18)	(22, 2, 4)
(5, 14, 15)	(0, 1, 7)	(6, 1, 4)	(14, 2, 12)	(2, 4, 20)	(17, 0, 11)	(11, 18, 1)
(22, 1, 17)	(20, 5, 13)	(16, 15, 13)	(1, 0, 4)	(21, 16, 5)	(14, 1, 21)	(0, 2, 14)
(5, 11, 6)	(14, 8, 9)	(2, 10, 15)	(2, 7, 19)	(18, 17, 16)	(2, 6, 14)	(2, 18, 1)
(9, 5, 2)	(16, 9, 4)	(14, 5, 0)	(11, 3, 8)	(1, 15, 8)	(1, 19, 17)	(9, 4, 1)
(5, 15, 8)	(13, 7, 3)	(21, 11, 7)	(15, 10, 4)	(2, 19, 10)	(19, 8, 7)	(5, 2, 3)
(7, 14, 20)	(2, 20, 7)	(1, 6, 10)	(5, 1, 20)	(4, 0, 7)	(5, 9, 14)	(17, 13, 21)
(15, 13, 2)	(21, 14, 16)	(1, 18, 14)	(11, 5, 9)	(6, 8, 5)	(12, 10, 1)	(2, 17, 6)
(11, 14, 0)	(12, 2, 8)	(9, 1, 10)	(9, 5, 17)	(11, 19, 16)	(3, 17, 2)	(5, 10, 19)
(12, 8, 11)	(3, 16, 4)	(5, 22, 10)	(13, 3, 1)	(4, 1, 0)	(3, 4, 7)	(7, 17, 18)
(14, 17, 3)	(7, 20, 14)	(8, 3, 5)	(0, 14, 4)	(4, 18, 2)	(22, 1, 8)	(13, 6, 10)
(17, 21, 7)	(7, 9, 1)	(1, 22, 11)	(15, 17, 13)	(10, 1, 9)	(16, 8, 18)	(12, 7, 4)
(16, 1, 15)	(7, 14, 21)	(0, 4, 5)	(3, 14, 5)	(16, 2, 21)	(5, 21, 16)	(1, 5, 19)
(1, 11, 15)	(11, 12, 17)	(4, 8, 19)	(1, 4, 6)	(20, 4, 8)	(18, 17, 7)	

Develop the initial $(18, \overrightarrow{P_3}, 2)$-ELGDD of type 6^3 under the automorphism group G_1 to get three $(18, \overrightarrow{P_3}, 2)$-ELGDDs of type 6^3, and develop the initial $(23, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}11^1$ under the automorphism group G_2 to get eighteen $(23, \overrightarrow{P_3}, 2)$-ELGDDs of type $1^{12}11^1$, all of which form a $(23, \overrightarrow{P_3})$-ELPCS of type $(6^3 : 5)$. □

4. Infinite families of LELDs

Now, we are in a position to establish several infinite classes for the existence of LELDs by recursion.

Lemma 4.1. *There exists an* $(n, \overrightarrow{P_3}, 2)$*-LELD for any integer* $n \geq 6$ *with* $n \equiv 2, 6, 14 \pmod{16}$ *or* $n \equiv 8 \pmod{12}$ *and* $n \neq 34, 50$.

Proof. For $n = 6, 8$, there is an $(n, \overrightarrow{P_3}, 2)$-LELD by Corollary 2.3.

For each $n = 16m+2, n = 16m+6$ or $n = 16m+14, n \geq 14$ and $n \neq 34, 50$, there is a 1-FG(3, ({3, 4, 5}, {4, 5, 6}), $(n-2)/4$) of type $1^{(n-2)/4}$, which is obtained by deleting one point from an S(3, {4, 5, 6}, $(n+2)/4$) (see [11]). Applying Theorem 2.7 with a $(4k-2, \overrightarrow{P_3})$-ELPCS of type $(4^{k-1} : 2)$ and an ELF($\overrightarrow{P_3}$, k{4}) with $k \in \{4, 5, 6\}$, we get an $(n, \overrightarrow{P_3})$-ELPCS of type $(4^{(n-2)/4} : 2)$. Then, applying Theorem 2.6 with a $(6, \overrightarrow{P_3}, 2)$-LELD, we obtain an $(n, \overrightarrow{P_3}, 2)$-LELD. Here, the input $(4k-2, \overrightarrow{P_3})$-ELPCSs of types $(4^{k-1} : 2)$ with $k \in \{4, 5, 6\}$ exist by Lemmas 3.9–3.11. The input ELF($\overrightarrow{P_3}$, 5{4}) and ELF($\overrightarrow{P_3}$, 6{4}) exist by Lemmas 3.6 and 3.7. The input ELF($\overrightarrow{P_3}$, 4{4}) is obtained by applying Theorem 2.5 with an H(2^4) and an ELF($\overrightarrow{P_3}$, 4{2}), which exist by Theorem 2.4 and Lemma 3.5, respectively.

For each $n = 12m-4$ and $m > 1$, there is a 1-FG(3, ({3, 5}, {4, 6}), $2m-1$) of type 1^{2m-1}, which is obtained by deleting one point from an S(3, {4, 6}, $2m$) (see [10]). Applying Theorem 2.7 with a $(6k-4, \overrightarrow{P_3})$-ELPCS of type $(6^{k-1} : 2)$ and an ELF($\overrightarrow{P_3}$, k{6}) with $k \in \{4, 6\}$, we get a $(12m-4, \overrightarrow{P_3})$-ELPCS of type $(6^{2m-1} : 2)$. Then, applying Theorem 2.6 with an $(8, \overrightarrow{P_3}, 2)$-LELD, we obtain an $(n, \overrightarrow{P_3}, 2)$-LELD. Here, the input $(6k-4, \overrightarrow{P_3})$-ELPCSs of types $(6^{k-1} : 2)$ with $k \in \{4, 6\}$ exist by Lemmas 3.12 and 3.13. The ELF($\overrightarrow{P_3}$, k{6}) with $k \in \{4, 6\}$ is obtained by applying Theorem 2.5 with an H(3^k) and an ELF($\overrightarrow{P_3}$, 4{2}). □

Lemma 4.2. *There exists an* $(n, \overrightarrow{P_3}, 2)$*-LELD for each positive integer* $n \equiv 11, 23 \pmod{36}$.

Proof. For $n = 11$, we obtain the design by applying Theorem 2.6 with a $(5, \overrightarrow{P_3}, 2)$-LELD and an $(11, \overrightarrow{P_3})$-ELPCS of type $(3^3 : 2)$. Simultaneously, we get an $(11, 5; \overrightarrow{P_3}, 2)$-HLELD.

For each $n = 36m+11$ or $n = 36m+23$ and $n \geq 23$, there is a 1-FG(3, (3, 4), $(n-5)/6$) of type $1^{(n-5)/6}$, which is obtained by deleting one point from an SQS($(n+1)/6$) (see [9]). Applying Theorem 2.7 with a $(23, \overrightarrow{P_3})$-ELPCS of type $(6^3 : 5)$ from Lemma 3.14 and an ELF($\overrightarrow{P_3}$, 4{6}), we get an $(n, \overrightarrow{P_3})$-ELPCS of type $(6^{(n-5)/6} : 5)$. Since there exists an $(11, 5; \overrightarrow{P_3}, 2)$-HLELD and an $(11, \overrightarrow{P_3}, 2)$-LELD, we obtain the desired $(n, \overrightarrow{P_3}, 2)$-LELD by Theorem 2.6. □

Combining Corollary 2.3, Lemmas 3.4, 4.1 and 4.2, we have the following theorem.

Theorem 4.3. *For each positive integer n,* $4 \leq n \leq 11$ *or* $n \geq 14$, $n \equiv k \pmod{144}$ *with* $k \in$ {2, 6, 8, 11, 14, 18, 20, 22, 23, 30, 32, 34, 38, 44, 46, 47, 50, 54, 56, 59, 62, 66, 68, 70, 78, 80, 82, 83, 86, 92, 94, 95, 98, 102, 104, 110, 114, 116, 118, 119, 126, 128, 130, 131, 134, 140, 142} *and* $n \neq 34, 50$, *there exists an* $(n, \overrightarrow{P_3}, 2)$*-LELD and an optimal, levelled* $(n-2)$*-fault-tolerant routing of* $\overrightarrow{K_n}$ *that has levelled minimum optical indices.*

5. Concluding remarks

As noted in [1–4,15], the design of fault-tolerant routings with levelled minimum optical indices has played an important role in the context of optical networks. Not much is known about the existence of optimal routings with levelled minimum optical indices besides the results established by Dinitz, Ling and Stinson [6] via the partitionable Steiner quadruple systems approach. However, very little has been established as regards the existence of partitionable Steiner quadruple systems despite much attention having been paid. It seems that the partitionable Steiner quadruple systems approach is not hopeful for giving a complete solution to the problem of constructing optimal routings with levelled minimum optical indices.

In this paper, we introduced the new concept of a large set of even levelled $\overrightarrow{P_3}$-designs. On the basis of the theory of three-wise balanced designs and partitionable candelabra systems, we proposed a new approach to constructing optimal routings with levelled minimum optical indices. Using this new method, we are able to give several infinite classes of routings having levelled minimum optical indices. We believe that our new approach will prove useful for solving the existence problem for optimal routings with levelled minimum optical indices.

Acknowledgements

The authors thank the anonymous referees and the Associate Editor for their helpful comments.

The second author's research was supported by the National Outstanding Youth Science Foundation of China under Grant No. 10825103, National Natural Science Foundation of China under Grant No. 10771193, Specialized Research Fund for the Doctoral Program of Higher Education, Program for New Century Excellent Talents in University, and Zhejiang Provincial Natural Science Foundation of China under Grant No. D7080064.

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